Continuous Variables Quantum Algorithm for solving Ordinary Differential Equations

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“Solving” ODEs

Given: \( t > 0, \ f : \mathbb{R}^d \rightarrow \mathbb{R}^d, x_0 \in \mathbb{R}^d \)

Return: \( x(t) \in \mathbb{R}^d \) such that:

\[
\begin{align*}
    x(0) &= x_0 \\
    \dot{x} &= f(x)
\end{align*}
\]

Problem 1: Initial Value Problem

Given: \( t > 0, \ f : \mathbb{R}^d \rightarrow \mathbb{R}^d, x_0 \in \mathbb{R}^d \)

Return: \( x(t) \in \mathbb{R}^d \) such that:

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Amplitude Encoded Solution Quantum Algorithms

Quantum algorithms with super-polynomial advantage return the solution as **amplitude encoded state**, complexity $\text{polylog}(d)$:

$$ |x\rangle = \frac{1}{\|x\|_2} \sum_{k=1}^{d} x_k |k\rangle $$

For these algorithms [1] proved “non-quantumness” overheads:

$$ O \left( e^{\delta(A) t} + \mu(A) \right) $$

Spectral gap

$$ \delta(A) = \text{Re}(\lambda_{\text{max}}(A)) - \text{Re}(\lambda_{\text{min}}(A)) $$

Orthonormality of eigenstates:

$$ \mu(A) = \|A^\dagger A - AA^\dagger\|^{1/2} $$


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Koopman Von Neuman (KvN) Formalism


• **Wavefunction** : Probability to be in a state at a given time
  \[ |\psi\rangle := \int_{-\infty}^{+\infty} \psi(x) |x\rangle_q dx \]
  \[ |\psi(x)|^2 = P(x(t) = x) \]

• **Hamiltonian** : Time evolution = Integration of ODE
  \[ \dot{x} = f(x) \rightarrow H = f(\hat{q})\hat{p} + \hat{p} f(\hat{q}) \]

Quantum Algorithms based on KvN

Different strategies to reduce infinite to finite Hilbert spaces:

- **Truncation** the Fock space [3]
- **Discretization** of the phase space [4]

\[
\Pi_k = \sum_{k=0}^{N} |k\rangle \langle k|
\]


Continuous Variables Quantum Algorithm

Set of **universal CV gates** in the infinite Hilbert space

What if we assume perfect universal CV Quantum Computation?

- Vacuum state
- Displacement operator
- Squeeze operator
- Rotation operator
- Cubic operator

**Flowchart:**

- Squeeze
- Displace
- Time Evolution
- Position Measurement

- Vacuum state
- Squeezed state
- Displaced state
- Time evolved state
- Measurement

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Goal: Solve the IVP with a given accuracy $\varepsilon$.

IVP $\rightarrow$ position eigenstate, which is a non-physical state “approximated” by squeezing the state vacuum with std $\sigma_0 = e^{-r}$

Given an upper bound on the Lyapunov exponent $\lambda$

The std of the final state $\sigma_f < \sigma_0 e^{\lambda t}$

Required initial squeezing power $r > \lambda t + \log(\varepsilon)$

NB: For non-linear dynamics, may be a biased estimator
Time Evolution, and Non-Gaussian Gates

\[ f(x) = \sum_{k=0}^{K} a_k x^k \rightarrow H = \sum_{k=0}^{K} a_k \{p, q^k\} \]

- Non-Linear ODE → Non-Gaussian Gates
- Trotterization: explicit procedure to compute gate sequence for one–dimensional polynomial ODE of degree K
- Preliminary scaling: \( \sim t 4^K \)

\[
\begin{align*}
  s &= \sqrt{t/(k+1)} \\
  r &= \sqrt{s/6} \\
  p^2, -s &\rightarrow q^{k+1}, -s \\
  p^2, +s &\rightarrow q^{k+1}, +s \\
  q^3, r &\rightarrow \{\hat{p}, \hat{q}^{k-1}\}, -r \\
  q^3, -r &\rightarrow \{\hat{p}, \hat{q}^{k-1}\}, -r
\end{align*}
\]
Initial Distribution Problem : Turning a bug into a feature

Problem 2 : Initial **Distribution** Problem

<table>
<thead>
<tr>
<th>Given :</th>
<th>$t &gt; 0, f : \mathbb{R}^d \to \mathbb{R}^d, p_0 : \mathbb{R}^d \to [0,1]$</th>
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<tbody>
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- Could be used to estimate Lyapunov exponents ?
- Could be used to characterize attractors ?
Thank you

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Questions?