Continuous Variables Quantum Algorithm for solving Ordinary Differential Equations

IEEE Quantum Week

Quantum Algorithms for Differential Equations Workshop

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"Solving" ODEs

$$\dot{x} = f(x)$$

Problem 1 : Initial **Value** Problem
Given : $t > 0, f: \mathbb{R}^d \to \mathbb{R}^d, x_0 \in \mathbb{R}^d$ Return : $x(t) \in \mathbb{R}^d$ such that : $x(0) = x_0$ $\dot{x} = f(x)$

Amplitude Encoded Solution Quantum Algorithms

Quantum algorithms with super-polynomial advantage return the solution as **amplitude encoded state**, complexity $\mathbf{polylog(d)}$:

$$|x\rangle = \frac{1}{\|x\|_2} \sum_{k=1}^{d} x_k |k\rangle$$

For these algorithms [1] proved "**non-quantumness**" overheads:

$$O\left(e^{\delta(A)t} + \mu(A)\right)$$

Spectral gap

$$\delta(A) = Re(\lambda)_{\max}(A) - Re(\lambda)_{\min}(A)$$

Orthonormality of eigenstates :

$$\mu(A) = \left\|A^\dagger A - AA^\dagger\right\|^{1/2}$$

[1] Dong An et al. "A theory of quantum differential equation solvers: limitations and fast-forwarding".

Koopman Von Neuman (KvN) Formalism

The framework [2] maps arbitrary classical dynamics to Hermitian dynamics in infinite Hilbert spaces.

- **Wavefunction** : Probability to be in a state at a given time

$$|\psi\rangle := \int_{-\infty}^{+\infty} \psi(x)\,|x\rangle_q\,dx$$

$$|\psi(x)|^2 = P(x(t) = x)$$

- **Hamiltonian** : Time evolution = Integration of ODE

$$\dot{x} = f(x) \rightarrow H = f(\hat{q})\hat{p} + \hat{p}f(\hat{q})$$

[2] B. O. Koopman, “Hamiltonian Systems and Transformation in Hilbert Space”

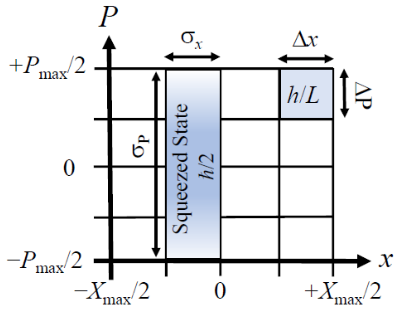
Quantum Algorithms based on KvN

Different strategies to reduce infinite to **finite Hilbert spaces** :

- **Truncation** the Fock space [3]

$$\Pi_k = \sum_{k=0}^{N} |k\rangle \langle k|$$

- **Discretization** of the phase space [4]

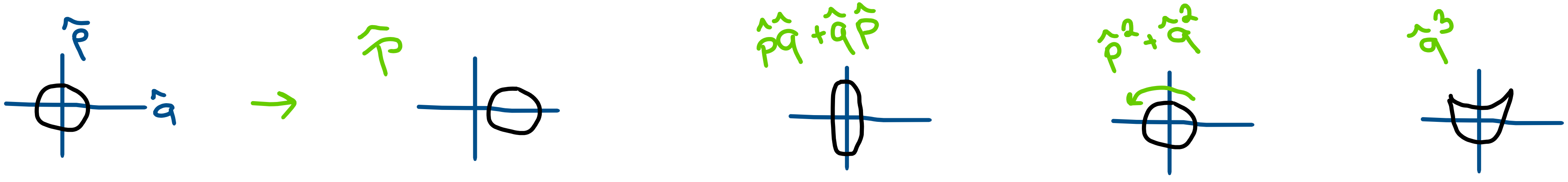


[3] A. Engel et al, "Linear embedding of nonlinear dynamical systems and prospects for efficient quantum algorithms"

[4] I. Joseph, "Koopman--von Neumann approach to quantum simulation of nonlinear classical dynamics"

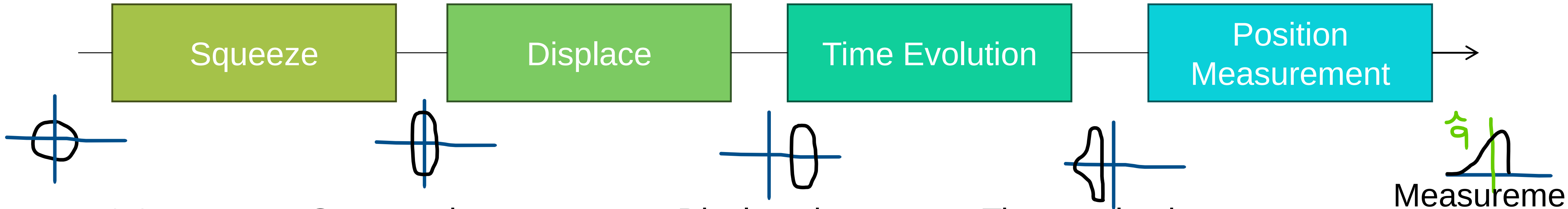
Continuous Variables Quantum Algorithm

Set of **universal CV gates** in the infinite Hilbert space



Vacuum state | Displacement operator | Squeeze operator | Rotation operator | Cubic operator

What if we assume perfect universal CV Quantum Computation ?



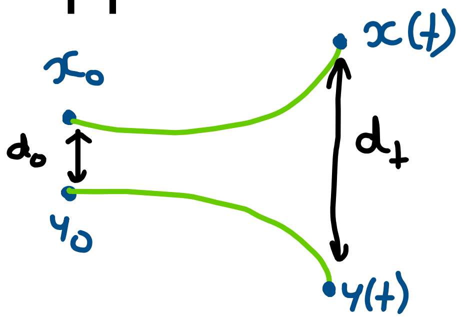
Vacuum state | Squeezed state | Displaced state | Time evolved state | Measurement

Initial State, Accuracy, Chaos and Squeezing

Goal : Solve the IVP with a given accuracy ϵ.

IVP → **position eigenstate**, which is a **non-physical** state "approximated" by **squeezing** the state vacuum with std $\sigma_0 = e^{-r}$

Given an upper bound on the **Lyapunov exponent λ**



$$\partial_t d = \lambda$$
$$d(t) = e^{\lambda t} d_0$$

The std of the final state $\sigma_f < \sigma_0 e^{\lambda t}$

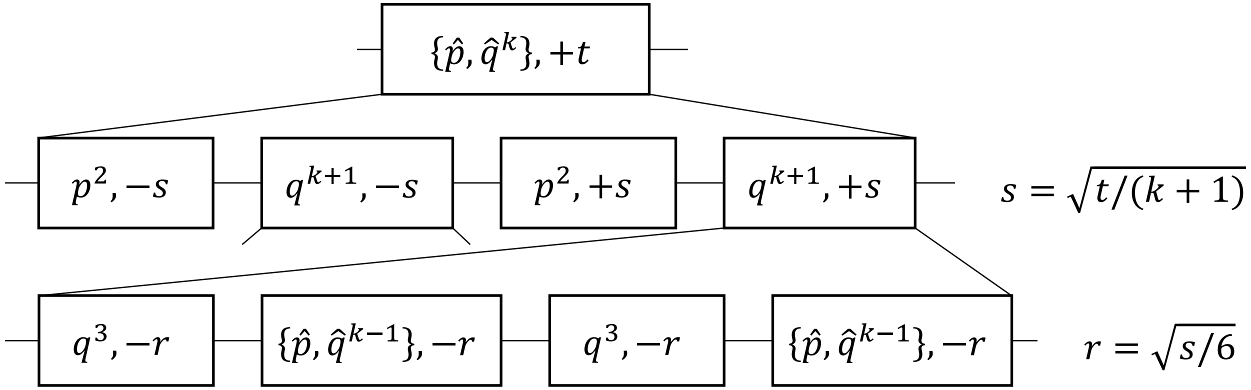
Required initial squeezing power $r > \lambda t + \log(\epsilon)$

NB : For non-linear dynamics, may be a **biased estimator**

Time Evolution, and Non-Gaussian Gates

$$f(x) = \sum_{k=0}^{K} a_k x^k \rightarrow H = \sum_{k=0}^{K} a_k \{p, q^k\}$$

- Non-Linear ODE → Non-Gaussian Gates
- Trotterization: **explicit procedure to compute gate sequence** for one–dimensional polynomial ODE of degree K
- Preliminary scaling : $\sim t\, 4^K$



Initial Distribution Problem : Turning a bug into a feature

Problem 2 : Initial **Distribution** Problem
Given : $t > 0, f: \mathbb{R}^d \to \mathbb{R}^d, p_0: \mathbb{R}^d \to [0,1]$ Return : $p_t: \mathbb{R}^d \to [0,1]$, such that : $\forall x\,, P(x(0) = x) = p_0(x)$ $\dot{x} = f(x)$ $\forall x\,, P(x(t) = x) = p_t(x)$

- Could be used to estimate Lyapunov exponents ?
- Could be used to characterize attractors ?

Thank you

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Questions ?